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ASYMPTOTIC ANALYSIS OF RECTILINEAR HYDRAULIC FRACTURING OF A PERMEABLE ELASTIC PLANE WITH SMALL AND LARGE FLUID LOSSES

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A rectilinear hydraulic fracture in a permeable elastic plane is one of three basic mechanical models used to design hydraulic fractures in productive reservoirs [1]. A simplified mathematical formulation of this model is employed in the present paper. The problem is reduced to a dimensionless system of equations for a single parameter. Asymptotics are considered for small and large values of the parameter; only the second one turned out to be known.

We consider a horizontal plane filled with a permeable elastic medium in which a symmetrical crack is growing under the action of a fluid pumped into the crack gap by the source at the center. The direction of crack growth is orthogonal to the initial compressing stresses P_g acting at infinity. The problem was first stated within the framework of fracture mechanics [2, 3]. An effective approximate solution for a crack growing in a permeable medium was found later [4]. Exact results for an impermeable and imponderable medium were obtained in [5]. The approach suggested in [5] is extended in the present paper. The problem is studied in a standard formulation without restrictions on the permeability. We assume that the fluid losses obey Carter's law [1, 6], i.e., the fluid volume equal to

$$\Lambda/\sqrt{T - T(X)} \qquad [T = T(R)]$$

is filtered off from a unit crack surface into the permeable medium per second. Here Λ is the leakage factor; T is the time reckoned from the beginning of pumping-in, at which the crack size runs to $2L$, and the boundary of the wetted section reaches the point $X = R$; and $T(X)$ is the time from which leakages began at distance X from the center.

The permeable medium can be impregnated by the fluid which is under a certain (reservoir) pressure. However, its back filtering into the dry region is considered negligible because of relatively small dimensions and displacement of the low-pressure zone as the crack grows. By virtue of symmetry, only the right half of the crack is considered. The X axis is directed along the crack, and the coordinate origin is placed at its center.

Let us define scales for the dimensionless variables so that the problem formulated below in dimensionless form depends on the least number of parameters. For this, we first introduce dimensional factors that are formed from the physical constants and parameters of the hydraulic fracturing process. Besides Λ and P_g, we shall need the elastic modulus E, the Poisson coefficient ν for an elastic medium, the viscosity μ of the fracturing fluid, and the flow rate Q_0 in each wing of the crack (the total flow rate $2Q_0$ is considered constant during pumping-in). We imply the flow rate in a real hydraulic fracture via a feed hole, which is distributed uniformly over the constant crack height H, and not the flow rate parameter $Q_* = Q_0/H$ in the two-dimensional problem. Below, all the flow-rate or bulk quantities introduced are referred to a unit height.

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Let us introduce the dimensional time T_c and length L_c factors:

$$T_c = 3\mu/D, \qquad D = 0.5E/(1-\nu^2); \tag{1}$$

$$L_c = \sqrt{0.5Q_* T_c}, \tag{2}$$

and choose

$$\sigma = P_g/D. \tag{3}$$

as a dimensionless characteristic of initial stresses.

Using the quantities given by formulas (1)–(3), we introduce the following scale factors for the cross-sectional and longitudinal crack dimensions, time, and "plane" volume, respectively:

$$W_* = \sigma^{-1}L_c, \quad L_* = \sigma^{-2}L_c, \quad T_* = \sigma^{-3}T_c, \quad \Omega_* = 2L_*W_* = Q_*T_*. \tag{4}$$

Denote the dimensional variables by capital letters and the dimensionless variables by corresponding lower-case letters. They are related to each other by

$$\begin{gathered} L = L_*l, \quad X = L_*x, \quad R = L_*r, \quad T = T_*t, \quad W(X) = W_*w(x), \quad P(X) = P_*p(x), \\ Q(X) = Q_*q(x), \quad \Omega_\rho = \Omega_*\omega_\rho, \quad \Omega_{\lambda\rho} = \lambda\Omega_*\omega_{\lambda\rho}, \quad \lambda = \Lambda L_*\sqrt{T_*}/\Omega_* = \Lambda/\sqrt{0.5Q_*\sigma}, \end{gathered} \tag{5}$$

where the dimensional variables $2W(X)$, $P(X)$, and $Q(X)$ are profiles of the fracture width, pressure, and local flow rate via the crack's cross-section; $2\Omega_\rho$ is the fluid volume in the crack; $2\Omega_{\lambda\rho}$ is the volume of fluid loss from the crack up to time T and λ is a characteristic of fluid loss that is formed from the process constants.

We begin to formulate equations with the law of conservation of mass. The flow rate Q through the cross-section that is at distance X from the center of the crack decreases monotonically with growing X. This is due to the fact that a part of the fluid is filtered off or fills the growing crack volume without reaching the point X. The corresponding balance is written as

$$Q + \frac{\partial\Omega}{\partial T} + \frac{\partial\Omega_\lambda}{\partial T} = Q_*. \tag{6}$$

Here

$$\Omega(X) = \int_0^X W(X')dX', \qquad \Omega(R) = \Omega_\rho; \tag{7}$$

$$\Omega_\lambda(X) = 2\Lambda\int_0^X dX'\sqrt{T-T'}, \qquad \Omega_\lambda(R) = \Omega_{\lambda\rho}, \qquad T' = T(X'). \tag{8}$$

The second term on the left-hand side of (6) characterizes the growth rate of the crack volume $\Omega(X)$ between the cross-sections with the 0 and X coordinates, and the third term characterizes the rate of fluid loss from this region, i.e., the growth rate of the volume of fluid loss $\Omega_\lambda(X)$.

Transforming to dimensionless variables in (6)–(8), according to (5), we obtain

$$q + \frac{\partial\omega}{\partial t} + \lambda\frac{\partial\omega_\lambda}{\partial t} = 1, \quad \omega(x) = \int_0^x w(x')dx', \quad \omega_\lambda(x) = 2\int_0^x dx'\sqrt{t-t'}. \tag{9}$$

For $x = r$ we write Eq. (9) in terms of total derivatives in the form $d\omega_\rho/dt + \lambda d\omega_{\lambda\rho}/dt = 1$ and, after integration with respect to t, we have

$$t = \omega_\rho + \lambda\omega_{\lambda\rho}, \quad \omega_\rho = \int_0^r w(x')dx', \quad \omega_{\lambda\rho} = 2\int_0^r dx'\sqrt{t-t'}. \tag{10}$$

For brevity, we shall formulate the remaining equations of the problem immediately in dimensionless form. In accordance with the results of fracture mechanics [7], for a normal fracture crack, the equation expressing the width of the crack in terms of the pressure gradient is written as

$$w = -\sqrt{l^2 - x^2} - l \int_0^r a(x, x') \frac{\partial p}{\partial x'} dx', \quad a(\varphi, \psi) = \pi^{-1}[2\psi \cos\varphi + b(\varphi, \psi)]; \tag{11}$$

$$b(\varphi, \psi) = (\sin\psi) \ln \frac{\cos\varphi + \cos\psi}{\cos\varphi - \cos\psi} - (\sin\varphi) \ln \frac{\sin(\varphi + \psi)}{\sin(\varphi - \psi)}. \tag{12}$$

The presence of the radical in the equation reflects the contribution of the initial stresses P_g to the dimensionless opening. According to [5], the kernel $a(\varphi, \psi)$ is written in the variables φ and ψ, which are introduced by the relations $x = l \sin\varphi$, $x' = l \sin\psi$, and $r = l \sin\rho$. Here the parameter ρ is additionally introduced. It characterizes the degree of filling of the crack by the fracturing fluid in angular units.

Another relation between the pressure gradient and the width in dimensionless variables, which follows from the hydrodynamic laws for laminar flow of a viscid fluid in a narrow crack, [2] can be written:

$$\partial p / \partial x = -q w^{-3}. \tag{13}$$

Thus, the problem is reduced to system (9)–(13). For analysis it is convenient to transform this system to new unknowns that would be asymptotically bounded as $\rho \to \pi/2$. For this, we first substitute (13) into (11) to eliminate the pressure p and transform from the linear to the angular variables φ and ψ. In addition, we move from w to the new function v:

$$w = v\sqrt{l} \cos\varphi. \tag{14}$$

The transformed equation takes the following form in terms of v:

$$v(\varphi) = -\sqrt{l} + \int_0^\rho \alpha(\varphi, \psi)\, q v^{-3} d\psi, \quad \alpha(\varphi, \psi) = a(\varphi, \psi)(\cos\varphi \cos^2\psi)^{-1}.$$

The counteraction to fracturing, i.e., to the failure of the elastic medium, is not yet reflected in the formulation of the problem. Since this has little effect on the growth of a large crack [4], the corresponding stress concentration at the crack tips is usually ignored, and smooth crack closure is assumed [2]. It can be demonstrated that, in terms of $v(\varphi)$, this is equivalent to the boundary condition

$$v(\pi/2) = 0. \tag{15}$$

The equation for v is consistent with (15) for the crack length:

$$l = \left[2\pi^{-1} \int_0^\rho \psi (\cos\psi)^{-2} q(\psi) v^{-3}(\psi) d\psi\right]^2. \tag{16}$$

In this case it can be written as an integral equation with power nonlinearity:

$$v(\varphi) = \int_0^\rho \beta(\varphi, \psi) q(\psi) v^{-3}(\psi) d\psi, \quad \beta(\varphi, \psi) = b(\varphi, \psi)(\pi \cos\varphi \cos^2\psi)^{-1}. \tag{17}$$

Formula (12) defines the function $b(\varphi, \psi)$. Condition (15) holds automatically by virtue of the properties of the kernel $\beta(\varphi, \psi)$.

In fact, the problem is reduced to a relatively simple integral equation only for a given flow rate. In the general case, however, the latter is an unknown and rather complex functional of v.

It is of interest to express the pressure in terms of the new variables. Integrating (13) with respect to x, we obtain the following relation in terms of v and l:

$$p(\varphi) = l^{-1/2} \int_\varphi^\rho q v^{-3} \cos^{-2}\psi d\psi.$$

If $p(\varphi)$ is averaged over the interval $[0, \pi/2]$, then, in view of (16), as would be expected with smooth closure [2, 4], the mean pressure coincides with the initial P_g, which is equal to unity in the chosen scale:

$$2\pi^{-1}\int_0^{\pi/2} p(\varphi)d\varphi = 2\pi^{-1}\int_0^{\rho} p(\varphi)d\varphi = 1.$$

The relative excess $\delta = p_0 - 1$ [$(p_0 = p(0)$] of the pressure at the center over the initial one is given by

$$\delta = \left[\int_0^{\rho}(\pi/2 - \psi)qv^{-3}\cos^{-2}\psi d\psi\right]\left(\int_0^{\rho}\psi qv^{-3}\cos^{-2}\psi d\psi\right)^{-1}. \tag{18}$$

We transform relations (9) for the flow rate. It is convenient to eliminate the time derivatives and write the flow rate in terms of $v(\varphi)$. Omitting cumbersome but simple calculations, we write the result as

$$q = 1 - \left(1 - \lambda l\int_0^{\rho}(t-t')^{-1/2}\cos\psi d\psi\right)s - \lambda l\int_0^{\varphi}(t-t')^{-1/2}\cos\psi d\psi, \tag{19}$$

where

$$s = s_\varphi/s_\rho, \quad s_\varphi = \int_0^{\varphi} v\left(\frac{3}{2} + \frac{l}{v}\frac{\partial v}{\partial l}\right)\cos^2\psi d\psi - v(\varphi, l)\sin\varphi\cos\varphi,$$
$$s_\rho = \int_0^{\rho} v\left(\frac{3}{2} + \frac{l}{v}\frac{\partial v}{\partial l}\right)\cos^2\psi d\psi + l\frac{d\rho}{dl}v(\rho, l)\cos^2\rho. \tag{20}$$

As will be seen below, the growth rate of a large crack is different for large and small fluid losses. It is expedient to take this fact into account explicitly in the equations. However, before proceeding to such a transformation, we introduce the functions

$$c(\gamma) = \begin{cases} 1 \\ \gamma^2 \end{cases}, \qquad d(\gamma) = \begin{cases} \gamma \\ 1 \end{cases}, \qquad \gamma = \lambda l^{1/4}, \tag{21}$$

which take the upper values for $\gamma < 1$ and the lower for $\gamma \geqslant 1$. The functions have inflections at the point $\gamma = 1$.

We further introduce the variables t and ω_ρ in place of τ and $\overline{\omega}_\rho$ by means of the relations

$$t = c(\gamma)\tau l^{3/2}, \quad t' = c(\gamma)\tau' l^{3/2}, \quad \omega_\rho = l^{3/2}\overline{\omega}_\rho; \tag{22}$$

$$\overline{\omega}_\rho = \int_0^{\rho} v(\psi)\cos^2\psi d\psi. \tag{23}$$

In the new variables, Eqs. (10) and (19) take the form

$$\tau = c^{-1}(\gamma)\overline{\omega}_\rho + 2d(\gamma)\int_0^{\rho}\sqrt{\tau - \tau'}\cos\psi d\psi; \tag{24}$$

$$q = 1 - \left(1 - d(\gamma)\int_0^{\rho}(\tau - \tau')^{-1/2}\cos\psi d\psi\right)s - d(\gamma)\int_0^{\varphi}(\tau - \tau')^{-1/2}\cos\psi d\psi. \tag{25}$$

Thus, the problem is reduced to system (16), (17), and (20)–(25). Solving this system, we find the width and pressure as functions of ρ, λ, and φ from formulas (14) and (18). We shall restrict ourselves to analytical results which can be obtained for small and large γ.

Let us analyze Eqs. (24) and (25). Assuming that v and τ are bounded, the relative order of the terms on the right sides of these equations is determined only by the coefficients $c(\gamma)$ and $d(\gamma)$. Proceeding from the structure of these coefficients, which is described by expressions (21), one can define a crack with weak or, conversely, very large fluid losses in terms of γ. For $\gamma \ll 1$, the contribution of integral terms to (24) and (25) which describe the fluid loss is small, and, therefore, the fluid loss can be considered weak. On the contrary, the case of $\gamma^{-2} \ll 1$ corresponds to great fluid loss. When $\gamma \ll 1$, the volume of fluid loss from the crack is small in comparison with the volume of the crack and affects the flow rate profile $q(\varphi)$ only slightly. When $\gamma^{-2} \ll 1$, most of the fluid goes into the permeable medium, and the fluid-loss rate forms the distribution $q(\varphi)$.

We estimate the characteristic value of λ using typical values of the determining dimensional constants. We assume that $E \cong 4 \cdot 10^{10}$ Pa, $\nu \cong 0.24$, $P_g \cong 4 \cdot 10^7$ Pa, $H \cong 25$ m, $Q_0 \cong 0.05$ m^3/sec, and $\Lambda \cong 10^{-4}$ m/sec$^{1/2}$. An estimate gives $\lambda \cong 5 \cdot 10^{-2}$. For this λ, the value of γ remains small, of the order of 0.1, even for $l \cong 100$; it becomes equal to unity only for $l \cong 1.6 \cdot 10^5$. To find the corresponding dimensional length of the crack L, we assume a typical viscosity value of $\mu \cong 0.05$ Pa $\cdot$ sec and, after estimating the scale factor L_* ($L_* = 7.5 \cdot 10^{-3}$ m), obtain $L \cong 1200$ m. A value of γ that is smaller by a factor of two will be reached for $L \cong 75$ m. Thus, for the chosen parameters of hydraulic fracture, the fluid loss begins to affect markedly only the growth of a fairly developed crack. Moreover, the case of $\gamma^{-2} \ll 1$ is actually unattainable. In contrast, the case of $\gamma \ll 1$ is of practical interest.

We now analyze the behavior of the crack with large sizes or, taking into account (15) and (16), with a high degree of filling, $\varepsilon = \pi/2 - \rho \to 0$. We consider first how the general system of equations is simplified when $\gamma \ll 1$. As it follows from (21) and (24), in a zeroth approximation for γ, the variable τ is expressed explicitly in terms of v:

$$\tau = \overline{\omega}_\rho = \int_0^\rho v(\psi) \cos^2\psi d\psi. \tag{26}$$

In this case, the volume of loss is defined by

$$\lambda\omega_{\lambda\rho} = 2\lambda \int_0^\tau dx' \sqrt{t - t'} = 2\lambda l^{7/4} \int_0^\rho \sqrt{\tau - \tau'} \cos\psi d\psi.$$

The remaining equations form the following system:

$$v(\varphi) = \int_0^\rho \beta(\varphi, \psi)\, q v^{-3} d\psi, \quad \sqrt{l} = 2\pi^{-1} \int_0^\rho \psi q v^{-3} \cos^{-2}\psi d\psi, \quad q = 1 - s_\varphi/s_\rho,$$

$$s_\varphi = \int_0^\varphi v\left(\frac{3}{2} + \frac{l}{v}\frac{\partial v}{\partial l}\right) \cos^2\psi d\psi - v(\varphi, l) \sin\varphi \cos\varphi, \tag{27}$$

$$s_\rho = \int_0^\rho v\left(\frac{3}{2} + \frac{l}{v}\frac{\partial v}{\partial l}\right) \cos^2\psi d\psi + l\frac{d\rho}{dl} v(\rho, l) \cos^2\rho.$$

Use of (27) is equivalent to neglect of fluid loss in the expression for the flow rate $q(\varphi)$. With the chosen procedure for rendering the equations dimensionless, (27) contains no function of physical constants, which were left in the scale factors. Thus, the solution is universal.

As the crack in the permeable medium grows, the condition $\gamma \ll 1$ is violated sooner or later, and the description of the hydraulic fracture based on system (27) becomes incorrect. However, in view of the weak dependence of $\gamma = \lambda l^{1/4}$ on l, this condition remains valid long enough for a poorly permeable medium. In this case, (27) can be further simplified by virtue of the asymptotic properties of the function v. On the one hand, from simple physical considerations, one should expect that $v(\varphi)$ would grow monotonically and level out with increasing l. On the other hand, analysis of the equation for v shows that v is bounded. Therefore, for large l the derivative $\partial v/\partial l$ must vanish almost everywhere except, probably, in a small vicinity of ρ.

The degree of filling ρ also increases monotonically with growing l and tends to the limit $\pi/2$. Then, analysis of the expressions for s_φ and s_ρ at large l shows that the contributions of all the derivatives with respect to l vanish. In this case, the problem becomes one-dimensional and reduces to the integral equation

$$v(\varphi) = \int_0^\rho \beta(\varphi, \psi)\, q(\psi)\, v^{-3}(\psi)\, d\psi; \tag{28}$$

$$q = \frac{\int_\varphi^\rho v \cos^2\psi d\psi + \frac{2}{3} v(\varphi, l) \sin\varphi \cos\varphi}{\int_0^\rho v \cos^2\psi d\psi}. \tag{29}$$

We attempt to demonstrate by simple estimates that the function v obtained by solution of this problem actually possesses properties that were used in deriving expression (29), i.e., v is bounded as ρ tends to $\pi/2$.

We divide the wetted section of the crack into two regions: a finite-size region, which almost reaches the fluid boundary, and a small region with a size of the order of the size ε of the dry region. The arguments and typical values of the functions for the first and second regions will be denoted by the subscripts 1 and 2, respectively.

We first estimate the quantities q_1 and q_2. We select sections in the integration intervals in (29) that correspond to the first and second regions. Taking into account that in the second region $\cos\psi \sim \varepsilon$, $\varepsilon = \pi/2 - \rho \ll 1$, we obtain $q_1 \sim 1$ and $q_2 \sim \varepsilon v_2/v_1$.

Next, we estimate the value of the kernel β in these regions. From the second formula of (17) we obtain the orders of magnitudes:

$$\beta(\varphi_1, \psi_1) \sim 1, \quad \beta(\varphi_1, \psi_2) \sim \varepsilon^{-1}, \quad \beta(\varphi_2, \psi_1) \sim \varepsilon, \quad \beta(\varphi_2, \psi_2) \sim \varepsilon^{-1}.$$

Substitution of these values together with the obtained estimates of q into (28) gives the order of v_1 and v_2:

$$v_1 \sim 1, \qquad v_2 \sim \varepsilon^{1/3}. \tag{30}$$

In particular, this indicates that v is bounded as $\varepsilon \to 0$.

Using (30), we correct the estimates of q:

$$q_1 \sim 1, \quad q_2 \sim \varepsilon^{4/3}. \tag{31}$$

Substituting (30) and (31) into Eq. 16, we obtain an asymptotic law for the crack growth as a function of the size ε of the dry region in angular units: $l \sim \varepsilon^{-4/3}$.

Let us derive asymptotic formulas for the other variables. We denote by ϖ_- the limit of ϖ_ρ for $\varepsilon \to 0$. According to formulas (22) and (26), we write the approximate relation $t \cong \omega_\rho \cong \varpi_- l^{3/2}$. Converting or, on inverting the function and transforming to dimensional variables, we obtain $L \cong 0.589\varpi_-^{-2/3}(D/\mu)^{1/6}Q_*^{1/2}T^{2/3}$. Thus, the dimensional length of the crack is asymptotically independent of the leakage factor and initial stresses.

The relative length $(l-r)/l$ of the dry region is also of interest. It decreases as the square of ε. In linear units, $l-r$ drops in inverse proportion to the square root of l:

$$l - r = (1 - \sin\rho)\, l \sim l\varepsilon^2 \sim l^{-1/2} \sim t^{-1/3}.$$

Let us consider the asymptotic behavior of the width of the crack. We denote by v_- the limiting value of v at the center of the crack when $\varepsilon \to 0$. By virtue of (14), we write approximately $w_0 \cong v_- l^{1/2} \cong v_- \varpi_-^{-1/3} t^{1/3}$, where w_0 is the dimensionless half-width at the center.

For the dimensional width $2W_0$ at the center of the crack we obtain the asymptotic expression

$$2W_0 \cong kP_g^{-1}\sqrt{\mu DQ_*}\sqrt{L}, \quad k = \sqrt{6}v_-\varpi_-^{-1/3}.$$

In the approximation in question, the width and length of the crack are independent of the leakage factor and initial stresses.

We also dwell on the asymptotic behavior of the pressure p_0 at the center of the crack. Turning to (18) and carrying out calculations similar to the above, we have $\delta = p_0 - 1 \sim l^{-1/2} \sim t^{-1/3}$. The asymptotic behavior of δ and $l - r$ is the same.

We finally give, for reference, an asymptotic formula that characterizes the growth in the volume of fluid loss $\lambda\omega_{\lambda\rho}$ with time:

$$\lambda\omega_{\lambda\rho} \cong 2\lambda\theta_- l^{7/4}, \quad \theta_- = \lim \int_0^\rho \sqrt{\tau - \tau'} \cos\psi d\psi \quad \text{for} \quad \rho \to \pi/2.$$

We analyze the crack growth in a well-permeable medium when the condition $\gamma^{-2} \ll 1$ is valid. In the zeroth approximation for γ^{-2}, Eqs. (24) and (25) assume the form

$$\tau = 2\int_0^\rho \sqrt{\tau - \tau'} \cos\psi d\psi, \quad q = 1 - \int_0^\varphi (\tau - \tau')^{-1/2} \cos\psi d\psi. \tag{32}$$

These relations split off from the remaining system. To find q and τ, we introduce the notation $\xi = \sin[\psi(\tau')]$ and $\eta = d\xi/d\tau'$ and rewrite (32):

$$\tau = 2\int_0^\tau \sqrt{\tau - \tau'}\eta(\tau')\, d\tau'.$$

Hence, using Laplace's transform, we find the function $\eta(\tau) = 1/(\pi\sqrt{\tau})$. Then, we have $\xi(\tau) = \sin\rho = 2\pi^{-1}\sqrt{\tau}$. Treating τ as a function of ρ, from this relation we obtain

$$\tau = \tau_+ \sin^2\rho, \quad \tau_+ = (\pi/2)^2. \tag{33}$$

The expression for the flow rate can now be written explicitly, and the entire problem is reduced to a one-dimensional integral equation in v, which does not contain dimensional physical parameters:

$$v(\varphi) = \int_0^\rho \beta(\varphi, \psi)\, qv^{-3} d\psi, \quad q = 1 - 2\pi^{-1} \arcsin(\sin\varphi / \sin\rho). \tag{34}$$

In moving to large lengths, the additional condition $l \gg 1$ or, what is the same, $\varepsilon \ll 1$, simplifies only slightly the form of the weight function q in this equation. With accuracy to terms of the order of ε^2, we write $q \cong 1 - 2\varphi/\pi$. Hence, we have estimates for q similar to (31) but for $\gamma^{-2} \ll 1$:

$$q_1 \sim 1, \qquad q_2 \sim \varepsilon. \tag{35}$$

In the same manner as for small γ, we obtain

$$v_1 \sim 1, \qquad v_2 \sim \varepsilon^{1/4}. \tag{36}$$

The limiting value of v_0 for $\varepsilon \to 0$ is denoted by v_+. Omitting calculations similar to those for small γ, we write the resulting asymptotic formulas for the behavior of the main variables for small ε and γ^{-2}:

$$\lambda l \cong \sqrt{t/\tau_+} \sim \varepsilon^{-3/2}, \quad L \cong \pi^{-1} Q_* \Lambda^{-1} T^{1/2},$$

$$2W_0 \cong 2v_+ \left(\frac{3\mu Q_*}{2D}\right)^{1/4} L^{1/2} \cong \frac{v_+}{\sqrt{\pi}} \left(\frac{24\mu}{D}\right)^{1/4} Q_*^{3/4} \Lambda^{-1/2} T^{1/4},$$

$$\omega_\rho \sim t^{3/4}, \quad \omega_{\lambda\rho} \sim t, \quad p_0 - 1 \sim l^{-1/2} \sim t^{-1/4}, \quad l - r \sim l^{-1/3} \sim t^{-1/6}.$$

These relations are well known [4] and are cited here mainly for completeness. It should be noted that they were obtained more rigorously, and this allowed us, in particular, to correct the exponents in the latter formula.

Let us compare the characteristic properties of the crack considered and of a crack of the related type described by the Perkins–Kern–Nordgren (PKN) model. Usually [1], attention is given only to opposing tendencies (drop and growth) in the pressure behavior $P(0)$, which are related to the degree of constriction for cracks of different types during their growth. Another, subtler difference is of interest from a mathematical point of view. When the problem in question is formulated in dimensionless form, the parameter $\lambda(\sigma)$ remains, while the PKN model, being properly made dimensionless, is entirely independent of the physical constants of the process [8], which are completely concentrated in the scale factors. In the PKN model, the initial pressure P_g is no more than the reference level for the pressure variable, while the value of P_g affects the behavior of the crack very significantly. The appearance of the dimensionless parameter $\lambda(\sigma)$ in the nonlocal model considered is due to the existence of a zero-pressure dry zone whose size depends on the initial pressure. At the same time, the form of the formulated system of equations suggests that the dry zone cannot be excluded from consideration by assuming "approximately" that, e.g., $\rho = \pi/2$. Although the proximity of ρ to $\pi/2$ is typical of a developed crack, it should be taken into account that, in essence, the degree of filling plays the role of time, and it cannot be considered constant if we wish to keep track of the process.

In conclusion, we note that the problem also remains nonlocal in the asymptotic cases considered. However, the absence of the dependence on λ is simply explained by the fact that these cases correspond to zero values of the small parameters γ or γ^{-2}.

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